Math 3280 Tutorial 8

Recall

① joint CDF & joint density:

$$F(a,b) = P(X \le a, Y \le b)$$
$$= \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\, dx\, dy$$

② For discrete X, Y, X, Y are independent

$$\Leftrightarrow p(x,y) = p_X(x) \cdot p_Y(y).$$

$p_X(x) \to P(X=x)$, $p_Y(y) \to P(Y=y)$

If X, Y are jointly cts, X, Y are independent

$$\Leftrightarrow f(x,y) = f_X(x) \cdot f_Y(y).$$

③ Independent cts r.v.s X, Y, $Z = X+Y$.

$$f_Z(a) = \int_{-\infty}^{\infty} f_X(a-y) \cdot f_Y(y)\, dy$$

Discrete r.v.s X, Y.

$$p(Z=k) = P(X+Y=k) = \sum_x p_X(x) \cdot p_Y(k-x)$$

Example 1: Let

$$f(x,y) = \begin{cases} \frac{1}{x}, & 0 < y < x < 1, \\ 0, & \text{o.w.} \end{cases}$$

(a) Show that $f(x,y)$ is a joint probability density function.

(b) Assume $f(x,y)$ is the joint density function of X, Y.

Find the densities of X, Y respectively.

Find $E[X]$, $E[Y]$.

Solution: (a).

$f(x,y)$ is non negative.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\left(\int_0^x \frac{1}{x}\,dy\right)dx$$

$$= \int_0^1 1\,dx$$

$$= \underline{1}$$

Hence $f(x,y)$ is a joint probability density function.

(b) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^x \frac{1}{x}\,dy$$

$$= \underline{1}, \quad \underline{x \in (0,1)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_y^1 \frac{1}{x}\,dx$$

$$= \ln x \Big|_{x=y}^{x=1}$$

$$= -\ln y, \quad \underline{y \in (0,1)}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_0^1 x \cdot 1\,dx$$

$$= \frac{1}{2}$$

$$E[Y] = \int_{\mathbb{R}} y \cdot f_Y(y)\,dy$$

$$= \int_0^1 y \cdot (-\ln y)\,dy$$

(integration by parts) $$= \int_0^1 -\ln y \; d\left(\frac{y^2}{2}\right)$$

$$= -\ln y \cdot \frac{y^2}{2} \Big|_{y=0}^{y=1} + \int_0^1 \frac{y^2}{2} \cdot \frac{1}{y}\,dy$$

$$= \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}.$$

Example 2: Suppose the joint probability mass function of X, Y are given by

$p(1,1)=\frac{1}{8}$, $p(1,2)=\frac{1}{4}$, $p(2,1)=\frac{1}{8}$, $p(2,2)=\frac{1}{2}$.

Are X, Y independent?

Solution:

$$P(X=1)=p(1,1)+p(1,2)$$
$$=\frac{1}{8}+\frac{1}{4}$$
$$=\frac{3}{8}$$

$$P(Y=1)=p(1,1)+p(2,1)$$
$$=\frac{1}{8}+\frac{1}{8}$$
$$=\frac{1}{4}$$

$$P(X=1,Y=1)=\frac{1}{8} \overset{?}{\neq} P(X=1)\cdot P(Y=1)=\frac{3}{8}\cdot\frac{1}{4}$$

So X, Y are not independent.

Example 3: If the joint density function of X, Y

$$f(x,y)=\begin{cases} 8xy, & 0<y<x<1 \\ 0, & \text{o.w.} \end{cases}$$

Are X, Y independent?

Solution:

cts X, Y. X, Y independent

$\Leftrightarrow f(x,y)=f_X(x)\cdot f_Y(y)$

$$f_X(x)=\int_{\mathbb{R}} f(x,y)\,dy = \int_0^x 8xy\,dy = 4x^3,\quad x\in(0,1)$$

$$f_Y(y)=\int_{\mathbb{R}} f(x,y)\,dx = \int_y^1 8xy\,dx = 4y\cdot x^2\Big|_{x=y}^{x=1}$$

$$=4y\cdot(1-y^2),\quad y\in(0,1).$$

$$f_X(x)\cdot f_Y(y) \neq f(x,y)$$

So X, Y are not independent.

4. The joint density function of X, Y is

$$f(x,y)=\begin{cases} x\cdot e^{-(x+y)}, & x>0, y>0 \\ 0, & \text{o.w} \end{cases}$$

Are X, Y independent?

Solution: $\Updownarrow$

$f(x,y)=f_X(x)\cdot f_Y(y)$.

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} x\cdot e^{-(x+y)}\,dy$$

$$= x e^{-x}\int_0^{\infty} e^{-y}\,dy \qquad x\le 0,$$

$$= x\cdot e^{-x}, \quad x>0. \qquad f_X(x)=0.$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} x\cdot e^{-(x+y)}\,dx$$

$$= e^{-y}\underbrace{\left(\int_0^{\infty} x\cdot e^{-x}\,dx\right)}_{=1}$$

expectation of exponential r.v. with $\lambda=1$

$$= e^{-y} \quad (y>0).$$

$$f_X(x)\cdot f_Y(y) = x\cdot e^{-(x+y)} \quad (\text{if } x>0, y>0)$$

$$f_X \cdots f_Y \cdots = \begin{cases} \text{[illegible]} & \text{[illegible]} \\ 0 & \text{o.w.} \end{cases}$$

So X, Y are independent r.v.s.